Final Exam — Partial Differential Equations (WBMA008-05)

Monday 16 June 2025, 15.00–17.00h

University of Groningen

Instructions

1. The use of calculators is *not* allowed. It is allowed to use a "cheat sheet" with notes (one sheet A4, both sides, handwritten,"wet ink").
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "1584" is not sufficient.
3. If p is the number of marks then the grade is $G = 1 + p/10$.

Problem 1 (10 + 5 + 5 = 20 points)

Consider the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + x^2 \frac{\partial u}{\partial x} = 0, \quad u(0,x) = \sin(\pi x).$$

(a) Compute all characteristic curves.

(b) Compute the value of the solution u at the point $(t,x) = (1,1)$.

(c) Is the solution u at the point $(t,x) = (1,-2)$ determined by the initial condition?

Problem 2 (12 + 8 = 20 points)

Consider the following damped wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial t}, \quad u(t,0) = u(t,\pi) = 0.$$

(a) Determine all real-valued nontrivial solutions of the form $u(t,x) = w(t)\sin(kx)$ with $k \in \mathbb{N}$.

(b) Derive a solution formula for the above equation with initial values $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ in terms of an infinite series and give expressions for the coefficients.

Problem 3 (15 points)

Compute Green's function for the following boundary value problem:

$$\frac{d}{dx}\left(x^2 \frac{du}{dx}\right) = f(x), \quad u(1) = 0, \quad u(2) = 0.$$

Please turn over for problems 4 and 5!

Problem 4 (5 + 10 = 15 points)

Consider the following Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y > 0,$$

with the boundary condition $u(x,0) = f(x)$.

(a) Show that $\widehat{u}(k,y) = \widehat{f}(k)e^{-y|k|}$ is a solution of the Fourier transformed equation.

(b) Determine a function $P_y(x)$ such that the solution of the boundary value problem can be written as $u(x,y) = (P_y * f)(x)$.

Problem 5 (5 + 15 = 20 points)

Consider Burgers' equation with $\gamma > 0$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \gamma\frac{\partial^2 u}{\partial x^2}.$$

(a) Show that substituting a travelling wave ansatz $u(t,x) = v(x-ct)$ leads to the ordinary differential equation

$$\gamma v' = \tfrac{1}{2}v^2 - cv + k,$$

where $k \in \mathbb{R}$ is an arbitrary constant.

(b) Assume $\gamma = c = \frac{1}{2}$ and $k = 0$. Compute the solution of the equation in part (a) that satisfies $v(0) = \frac{1}{2}$ and compute the corresponding solution $u(t,x)$.

Please do not forget to complete the course evaluation!

End of test (90 points)

Solution of problem 1 (10 + 5 + 5 = 20 points)

(a) The characteristic curves $t \mapsto (t, x(t))$ are found by solving the following ordinary differential equation:

$$\frac{dx}{dt} = x^2.$$

(2 points)

Note that the curve $t \mapsto (t, 0)$, i.e. the line $x = 0$, is a characteristic curve.
(2 points)

To find the remaining characteristic curves, we use separation of variables:

$$\int \frac{1}{x^2}\,dx = \int dt \quad \Rightarrow \quad -\frac{1}{x} = t + k \quad \text{or} \quad x = -\frac{1}{t+k}.$$

(6 points)

(b) The point $(t, x) = (1, 1)$ lies on the characteristic curve given for $k = -2$.
(2 points)

This characteristic curve intersects the x-axis in the point $(0, \frac{1}{2})$.
(2 points)

Since the points $(1, 1)$ and $(0, \frac{1}{2})$ lie on the same characteristic curve and the solution u is constant along such a curve, we have

$$u(1,1) = u(0, \tfrac{1}{2}) = \sin(\tfrac{1}{2}\pi) = 1.$$

(1 point)

(c) The point $(t, x) = (1, -2)$ lies on the characteristic curve given for $k = -\frac{1}{2}$.
(2 points)

Note that the equation

$$x = -\frac{1}{t - 1/2}$$

actually specifies *two distinct curves* in the (t, x)-plane, namely one branch for $t > 1/2$ and another branch for $t < 1/2$. The branch that contains the point $(1, -2)$ does not intersect the x-axis. Therefore, the solution at the point $(t, x) = (1, -2)$ is not determined by the initial condition.
(3 points)

Solution of problem 2 (12 + 8 = 20 points)

(a) Substituting the ansatz $u(t,x) = w(t)\sin(kx)$ into the partial differential equation gives the following ordinary differential equation:

$$w''(t) = -4k^2 w(t) - 2w'(t) \quad \Leftrightarrow \quad w''(t) + 2w'(t) + 4k^2 w(t) = 0.$$

(3 points)

Setting $w(t) = e^{\lambda t}$ gives the characteristic equation

$$\lambda^2 + 2\lambda + 4k^2 = 0.$$

(3 points)

Since the discriminant $4 - 16k^2$ is negative for all $k \in \mathbb{N}$, the roots are complex:

$$\lambda = -1 \pm \omega_k i \quad \text{where} \quad \omega_k = \sqrt{4k^2 - 1}.$$

(3 points)

Therefore, the non-trivial solutions for u are given by

$$u(t,x) = e^{-t}\cos(\omega_k t)\sin(kx) \quad \text{and} \quad u(t,x) = e^{-t}\sin(\omega_k t)\sin(kx).$$

(3 points)

(b) By superposition we obtain the infinite series

$$u(t,x) = e^{-t}\sum_{k=1}^{\infty}\left[a_k\cos(\omega_k t) + b_k\sin(\omega_k t)\right]\sin(kx).$$

The initial condition $u(0,x) = f(x)$ implies

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(kx)$$

which gives

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx.$$

(4 points)

The initial condition $u_t(0,x) = g(x)$ implies

$$g(x) = \sum_{k=1}^{\infty}\left[-a_k + b_k\omega_k\right]\sin(kx)$$

which gives

$$b_k = \frac{a_k}{\omega_k} + \frac{2}{\pi\omega_k}\int_0^{\pi} g(x)\sin(kx)\,dx.$$

(4 points)

Solution of problem 3 (15 points)

First, solve the homogeneous equation:

$$\frac{d}{dx}\left(x^2\frac{du}{dx}\right) = 0 \quad \Rightarrow \quad x^2\frac{du}{dx} = a$$
$$\Rightarrow \quad \frac{du}{dx} = \frac{a}{x^2}$$
$$\Rightarrow \quad u(x) = b - \frac{a}{x},$$

where $a, b \in \mathbb{R}$ are arbitrary constants.
(5 points)

Note that:

- $u(x) = 1 - 1/x$ satisfies the homogeneous equation and the condition $u(1) = 0$;
- $u(x) = 1 - 2/x$ satisfies the homogeneous equation and the condition $u(2) = 0$.

Therefore, our Green's function is of the form

$$G(x;\xi) = \begin{cases} a(1-1/x) & \text{if } x \leq \xi, \\ b(1-2/x) & \text{if } x \geq \xi. \end{cases}$$

(3 points)

Requiring that G is continuous at $x = \xi$ implies that

$$a\left(1 - \frac{1}{\xi}\right) - b\left(1 - \frac{2}{\xi}\right) = 0.$$

(2 points)

Requiring that $\partial G/\partial x$ makes a jump of magnitude $1/p(\xi) = 1/\xi^2$ implies

$$\frac{2b}{\xi^2} - \frac{a}{\xi^2} = \frac{1}{\xi^2}.$$

(2 points)

Solving these equations gives

$$G(x;\xi) = \begin{cases} (1-2/\xi)(1-1/x) & \text{if } x \leq \xi, \\ (1-1/\xi)(1-2/x) & \text{if } x \geq \xi. \end{cases}$$

(3 points)

Solution of problem 4 (5 + 10 = 15 points)

(a) Taking the Fourier transform with respect to the x variable gives

$$-k^2\widehat{u}(k,y)+\widehat{u}''(k,y)=0,$$

where the primes denote differentiation with respect to y.
(3 points)

Setting $\widehat{u}(k,y)=\widehat{f}(k)e^{-y|k|}$ and differentiating with respect to y twice gives

$$\widehat{u}''(k,y)=(-|k|)^2\widehat{f}(k)e^{-y|k|}=k^2\widehat{u}(k,y),$$

which shows that the given function is indeed a solution.
(2 points)

(b) From the table of Fourier transforms we obtain

$$\mathscr{F}\left[e^{-a|x|}\right]=\sqrt{\frac{2}{\pi}}\frac{a}{k^2+a^2}.$$

(3 points)

The symmetry principle gives

$$\mathscr{F}\left[\sqrt{\frac{2}{\pi}}\frac{a}{x^2+a^2}\right]=e^{-a|-k|}=e^{-a|k|}.$$

(3 points)

By taking $a=y$ and setting

$$P_y(x)=\frac{1}{\pi}\frac{y}{x^2+y^2}$$

part (a) gives that

$$\widehat{u}(k,y)=\sqrt{2\pi}\,\widehat{P}_y(k)\widehat{f}(k)=\widehat{(P_y*f)}(k)$$

and thus $u(x,y)=(P_y*f)(x)$.
(4 points)

Solution of problem 5 (5 + 15 = 20 points)

(a) Substituting the travelling wave ansatz $u(t,x) = v(x-ct)$ into Burgers' equation gives

$$-cv' + vv' = \gamma v''.$$

(2 points)

Noting that $vv' = (\frac{1}{2}v^2)'$ and integrating both sides gives

$$k - cv + \tfrac{1}{2}v^2 = \gamma v',$$

where $k \in \mathbb{R}$ is an arbitrary constant.
(3 points)

(b) Assuming that $\gamma = c = \frac{1}{2}$ and $k = 0$ gives

$$v' = v^2 - v.$$

Separation of variables gives

$$\begin{aligned} \int \frac{1}{v(v-1)}\,dv = \int d\xi \quad &\Rightarrow \quad \int \frac{1}{v-1} - \frac{1}{v}\,dv = \int d\xi \\ &\Rightarrow \quad \log|v-1| - \log|v| = \xi + \delta. \end{aligned}$$

(4 points)

The solution that satisfies $v(0) = \frac{1}{2}$ lies between the constant solutions $v = 0$ and $v = 1$ and we obtain

$$\begin{aligned} \log|v-1| - \log|v| = \xi + \delta \quad &\Rightarrow \quad \log(1-v) - \log(v) = \xi + \delta \\ &\Rightarrow \quad \log\left(\frac{1-v}{v}\right) = \xi + \delta \\ &\Rightarrow \quad \frac{1-v}{v} = e^{\xi+\delta} \\ &\Rightarrow \quad v(\xi) = \frac{1}{1+e^{\xi+\delta}}. \end{aligned}$$

(4 points)

In addition, the condition $v(0) = \frac{1}{2}$ gives $\delta = 0$. Finally, we obtain

$$u(t,x) = v(x - t/2) = \frac{1}{1+e^{x-t/2}}.$$

(2 points)